Journal of Applied Mechanics and Technical Physics, Vol. 41, No. 3, 2000

ESTIMATE OF THE STRENGTH OF A PLATE WITH AN ELLIPTIC HOLE IN TENSION AND COMPRESSION

S. V. Suknev

UDC 539.4

The strength of a plate with an elliptic hole under uniaxial tension or compression is estimated for arbitrary angles between the ellipse axes and the direction of loading with the use of the gradient strength criterion. The calculated critical stress agrees with the existing experimental data.

Formulation of the Problem. We consider a thin isotropic homogeneous linear-elastic plate of a brittle material loaded by the tensile σ or compressive p stresses distributed uniformly at infinity. There is an elliptic hole at the center of the plate which is oriented at an angle ω to the direction of loading (Fig. 1; a and b are the major and minor semiaxes of the ellipse). We determine the critical stress $\sigma_c(p_c)$ at which the limit state (local fracture) occurs in the plate. The averaged mechanical properties of the plate material are characterized by the ultimate stress σ_0 determined for a hole-free plate.

In accordance with the traditional approach to strength analysis, σ_0 is assumed to be a constant of the material and the strength condition is written in the form

$$\sigma_e < \sigma_0. \tag{1}$$

Here $\sigma_e = f(\sigma_{ij})$ is the equivalent stress which characterizes the internal stresses in the body and is a function of the stress-tensor components σ_{ij} in the general case. The critical stress is estimated as follows:

$$\sigma_c = \sigma_0 / K_t. \tag{2}$$

Here K_t is the stress-concentration coefficient which characterizes the ratio of the equivalent stress σ_e at the most stressed point of the body to the applied stress σ.

The range of applicability of expression (2) is restricted to the case of small K_t, where the dimension of the stress-nonuniformity zone is sufficiently large to assume that $\sigma_0 = \text{const}$. As applied to our case, this means that expression (2) can be used only for small angles ω. As ω increases, the stress-concentration coefficient rapidly increases, and the error of determining the critical stress from formula (2) becomes pronounced, which is supported by the experimental data of [1].

To extend the range of applicability of expression (2), some researchers [2–5] suggested using the stress condition (1) at the point of maximum equivalent stress located on the external contour that lies at a certain distance from the hole rather than on the hole boundary. In this connection, the following two questions arise: what is the shape of the external contour and how far is it from the hole boundary? Kipp and Sih [2] and Maiti and Smith [4] used a circle of radius r whose center is located at the point of maximum shear stress on the hole boundary, and Wu and Chang [3] and Yeh and Kim [5] used the confocal ellipse with the major semiaxis $a + r$ as an external contour. The value of r is chosen for consistency of calculation results with the experimental data. The critical stress was calculated with the use of different strength criteria: the maximum

Institute of Physicotechnical Problems of the North, Siberian Division, Russian Academy of Sciences, Yakutsk 677891. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 41, No. 3, pp. 163–168, May–June, 2000. Original article submitted July 20, 1999.

 0021-8944/00/4103-0526 $25.00

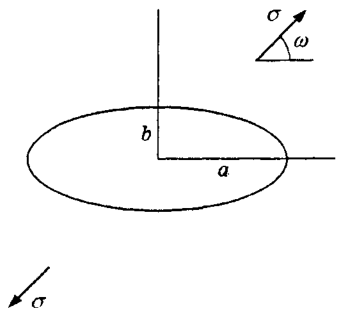


Fig. 1

shear stress and the maximum principal shear stress [4], the maximum principal stress [4, 5], the maximum shear strain [3, 4], and the minimum density of strain energy [2, 4]. Comparison between the calculated values of σ_c and the experimental data has not revealed the most adequate strength criterion [4]. Evidently, this is due to the fact that the approximation parameter r is not associated with the physical mechanisms of material fracture.

Application of the Gradient Strength Criterion. Suknev [6] proposed another approach in which the hypothesis $\sigma_0 = \text{const}$ was abandoned. The strength condition is written in the form $\sigma_e < f(\sigma_0, L_0/L_e)$, and the critical stress is determined by the expression $\sigma_c = \min\{f(\sigma_0, L_0/L_e)/(\sigma_e/\sigma)\} > 0$, where L_0 is the characteristic dimension of the material structure and L_e is the characteristic dimension of the deformable region. In the presence of stress concentration, L_e is determined by the dimension of the stress-nonuniformity zone. If this dimension is sufficiently large compared to the dimensions of the structural components of the material, including admissible defects of the structure (i.e., the conditions of averaging the mechanical properties are fulfilled), the value of the local strength differs slightly from σ_0. If L_e is comparable to the dimensions of the structural elements, their influence on the local strength becomes noticeable; it is the more pronounced the smaller the ratio L_e/L_0. To estimate L_0 and L_e, one can use the critical dimension of the defect $l_c = 2K_c^2/(\pi\sigma_0^2)$ (K_c is the critical stress-intensity coefficient) and the curvature radius of a concentrator at the "dangerous" point ρ, respectively.

With allowance for these estimates, the local-strength function for the main problem of a plate with an elliptic hole ($\omega = 0$ and $90°$) in symmetric tension has the form [6]

$$f(\sigma_0, L_0/L_e) = \sigma_0(1 + \sqrt{2l_c/\rho}). \tag{3}$$

We now extend function (3) to arbitrary angles $0 \leqslant \omega \leqslant 90°$ and values of $0 \leqslant m < 1$, where $m = (a-b)/(a+b)$ is the geometrical parameter, assuming that upon tension and compression a detachment-like failure occurs, which is determined by normal stresses, i.e., $\sigma_e = \sigma_\theta > 0$ (σ_θ is the shear stress on the hole contour).

Tension. The problem of σ_c determination is to find the minimum

$$\sigma_c = \min\left\{\sigma_0 \frac{1 + m^2 - 2m\cos 2\theta + \sqrt{2l_c(1-m)/a}\,(1+m)(1 + m^2 - 2m\cos 2\theta)^{1/4}}{1 - m^2 + 2m\cos 2\omega - 2\cos(2\theta - 2\omega)}\right\}, \quad \sigma_c > 0. \tag{4}$$

Here the well-known expression for the stress on the contour of an elliptic hole σ_θ [7] and the expression for the curvature radius of the contour $\rho_\theta = a(1 + m^2 - 2m\cos 2\theta)^{3/2}/[(1+m)^2(1-m)]$ were used.

The varied parameter in expression (4) is the angle θ, which satisfies the equation

$$A - B\sin 2\theta - C\cos 2\theta - F(\theta) = 0, \tag{5}$$

where $A = 2m\sin 2\omega$, $B = (1-m^2)(m - \cos 2\omega)$, $C = (1+m^2)\sin 2\omega$, and $F(\theta) = \sqrt{l_c/(2\rho_\theta)}\,[0.5m\sin 2\theta(1 - m^2 + 2m\cos 2\omega - 2\cos(2\theta - 2\omega)) - 2(1 + m^2 - 2m\cos 2\theta)\sin(2\theta - 2\omega)]$.

We solve Eq. (5) by the method of successive approximations:

$$A - B\sin 2\theta^{(k)} - C\cos 2\theta^{(k)} = F(\theta^{(k-1)}), \qquad k = 1, 2, \ldots,$$
$$\sin 2\theta^{(k)} = \frac{A^{(k)}B - C\sqrt{B^2 + C^2 - (A^{(k)})^2}}{B^2 + C^2}, \quad \cos 2\theta^{(k)} = \frac{A^{(k)}C + B\sqrt{B^2 + C^2 - (A^{(k)})^2}}{B^2 + C^2}, \tag{6}$$

where $A^{(k)} = A - F(\theta^{(k-1)})$ and $A^{(0)} = A$.

For $l_c < 2\rho_\theta$, the iteration process (6) rapidly converges to provide the required accuracy in determining the critical stress. Thus, for $l_c/a = 0.01$ and $m = 2/3$, one iteration is sufficient to determine σ_c with an accuracy of 0.1% over the entire range of variation of the angle ω. If $l_c \gg 2\rho_\theta$, the iteration procedure fails. In practice, this corresponds to the case of a narrow ellipse and large values of ω where σ_c remains almost unchanged as ω increases. Therefore, in determining the "dangerous" point, we can restrict our analysis to the zeroth approximation in (6).

We now consider the case where $m \to 1$. The elliptic hole becomes a cut. At the same time, the real crack has a finite radius of curvature at the tip. In the experiment, only its length can be measured (with allowance for slow crack extension before the unstable growth). Therefore, we have a hole for which a is known and ρ is unknown. Since ρ is merely the estimating parameter for the characteristic dimension L_e, the outlined procedure for determining the critical stress can be substantially simplified.

For the given angled ellipse, we consider an equivalent symmetric ellipse with the parameters $K_t = K_{t\omega}$ and $a_e = a_\omega$ which have the form

$$K_{t\omega} = \frac{1 - m^2 + 2m\cos 2\omega - 2\cos(2\theta - 2\omega)}{1 + m^2 - 2m\cos 2\theta}, \qquad a_\omega = a\,\frac{(1 + m^2 + 2m\cos 2\theta)^{1/2}}{1 + m}. \tag{7}$$

Here $K_{t\omega}$ is the stress-concentration coefficient of the angled ellipse, $2a_\omega$ is the dimension of the angled ellipse in the "dangerous" cross section which passes through the points of stress concentration, and $2a_e$ is the dimension of the equivalent ellipse in the "dangerous" cross section. The angle θ in expressions (7) is determined by solving Eq. (6) in the zeroth approximation. In this case, the radius of curvature at the vertex of the equivalent ellipse ρ_e is estimated by the formula

$$\rho_e = 4a_\omega/(K_{t\omega} - 1)^2, \tag{8}$$

which follows from the expression for the stress-concentration coefficient for the symmetric ellipse $K_t = 1 + 2\sqrt{a_e/\rho_e}$.

Now we can use the formula for estimation of the critical stress upon symmetric tension [6]:

$$\sigma_c = \sigma_0(1 + \sqrt{2l_c/\rho_e})/K_t. \tag{9}$$

Thus, the initial strength problem of a plate in tension for $\omega \neq 0$ and 90° reduces to the equivalent symmetric problem ($\omega = 90°$) with local-strength function (3) and estimate (8) for the characteristic dimension L_e.

Compression. The critical stress is determined as $p_c = \min\{-f(\sigma_0, L_0/L_e)/(\sigma_\theta/p)\} > 0$. Using the same expressions for σ_θ and ρ_θ as in the case of tension, we obtain

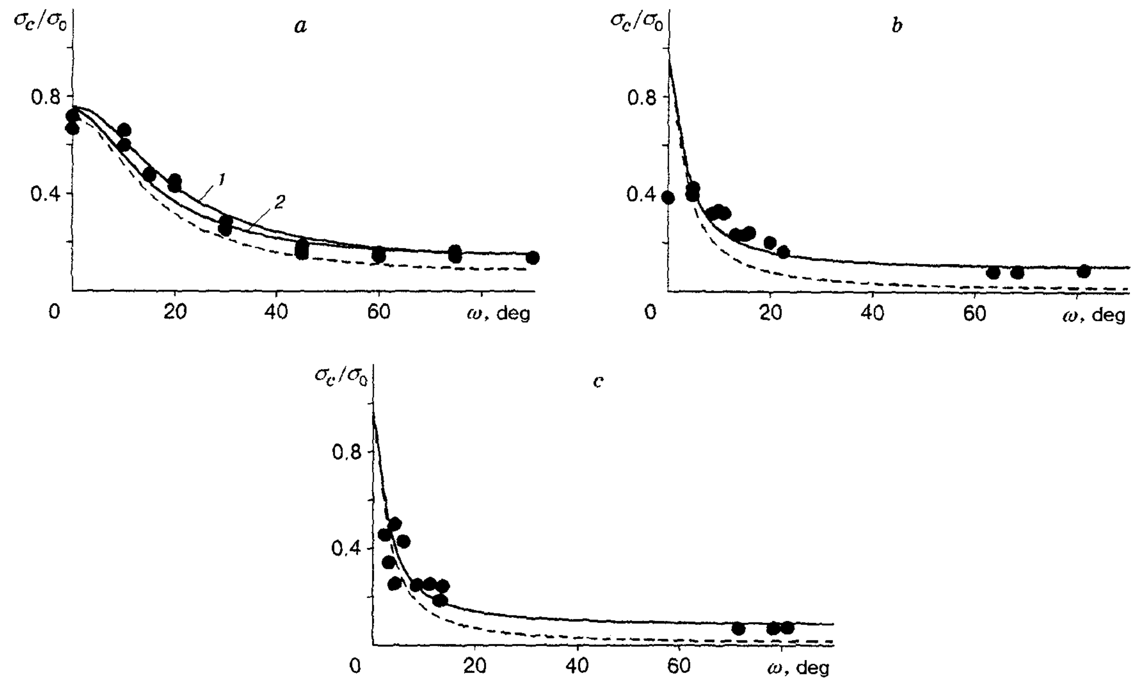
$$p_c = \min\left\{ -\sigma_0\,\frac{1+m^2 - 2m\cos 2\theta + \sqrt{2l_c(1-m)/a}\,(1+m)(1 + m^2 - 2m\cos 2\theta)^{1/4}}{1 - m^2 + 2m\cos 2\omega - 2\cos(2\theta - 2\omega)} \right\}, \quad p_c > 0. \tag{10}$$

The angle θ is determined by solving Eq. (5) with the use of the iteration procedure (6):

$$\sin 2\theta^{(k)} = \frac{A^{(k)}B + C\sqrt{B^2 + C^2 - (A^{(k)})^2}}{B^2 + C^2}, \quad \cos 2\theta^{(k)} = \frac{A^{(k)}C - B\sqrt{B^2 + C^2 - (A^{(k)})^2}}{B^2 + C^2}.$$

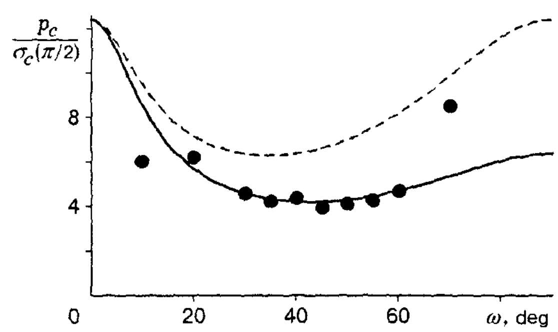
Here $A^{(k)} = A - F(\theta^{(k-1)})$ and $A^{(0)} = A$.

Comparison between Theoretical and Experimental Data. The expressions for the critical stress (4), (9), and (10), which were obtained on the basis of the gradient criterion, were used to estimate the strength of polymethylmethacrylate (PMMA) plates with elliptic holes subjected to uniaxial tension or compression.



σ_c/σ_0
a
b
c
0.8
0.4
0
20
40
60
ω, deg
1
2

Fig. 2



$\frac{p_c}{\sigma_c(\pi/2)}$
8
4
0
20
40
60
ω, deg

Fig. 3

Wu, Yao, and Yip [1] tested tensioned plates 380 mm long, 152 mm wide, and 3.2 mm thick. The semiaxes of the elliptic hole were $a = 12.7$ mm and $b = 2.5$ mm. The critical load was determined at the instant the specimen began to fail, which occurred suddenly. Figure 2a shows experimental data (points) and the critical stress calculated by formulas (4) and (9) (curves 1 and 2, respectively) and formula (2) (the dashed curve). It is noteworthy that the calculation by formula (9) is simpler compared to that by formula (4).

Williams and Ewing [8] tested tensioned plates 305 mm long, 152 mm wide, and 3.2 mm thick. The holes were shaped like narrow slits with ends notched by a razor blade. The critical load was determined at the instant the specimen began to fail. In the process, the specimens in which the crack extension was no greater than 0.3 mm before the onset of unstable growth (fracture) were used. Figure 2b (for $a = 12.7$ mm) and c (for $a = 17.8$ mm) shows experimental data (points) and the critical stress calculated by formula (9) (solid curves). One can see that the theoretical results agree with the experimental data. Since the radius of curvature at the notch tip was not measured, the value of $\rho = 0.02$ used in the experiments was obtained by approximating the experimental data. The dashed curves refer to calculation results by formula (2).

Tirosh and Catz [9] tested 8-mm-thick compressed square plates with a side of 50 mm. The holes made by a CO_2 laser were shaped like slits of length 14 mm and roundness diameter 0.43 mm at the tip. The experimental dependence of the critical compressive stress $p_c(\omega)$ normalized to the critical tensile stress σ_c determined for $\omega = 90°$ is shown in Fig. 3 by dots. Figure 3 also shows the ratio $p_c(\omega)/\sigma_c(\pi/2)$ calculated by formulas (10) and (4) (the solid curve), which agrees with the experimental data (points). The dashed curve was obtained with the use of criterion (1).

Thus, the local-strength function (3) determined on the basis of the gradient approach [6] can be used to estimate the strength of a plate with an elliptic hole both in tension and compression. This is supported by comparison between the predicted critical stress and the existing experimental data obtained on PMMA specimens.

REFERENCES

1. H.-C. Wu, R. F. Yao, and M. C. Yip, "Experimental investigation of the angled elliptic notch problem in tension," *Trans. ASME. Ser. E, J. Appl. Mech.*, **44**, No. 3, 455–461 (1977).
2. M. E. Kipp and G. C. Sih, "The strain energy density failure criterion applied to notched elastic solids," *Int. J. Solids Struct.*, **11**, No. 2, 153–173 (1975).
3. H.-C. Wu and K.-J. Chang, "Angled elliptic notch problem in compression and tension," *Trans. ASME, Ser. E, J. Appl. Mech.*, **45**, No. 2, 258–262 (1978).
4. S. K. Maiti and R. A. Smith, "Comparison of the criteria for mixed mode brittle fracture based on the preinstability stress–strain field. Part I. Slit and elliptical cracks under uniaxial tensile loading," *Int. J. Fract. Mech.*, **23**, No. 4, 281–295 (1983).
5. H.-Y. Yeh and C. H. Kim, "Fracture mechanics of the angled elliptic crack under uniaxial tension," *Eng. Fract. Mech.*, **50**, No. 1, 103–110 (1995).
6. S. V. Suknev, "Application of a gradient approach to estimation of the local strength," *Prikl. Mekh. Tekh. Fiz.*, **40**, No. 4, 222–228 (1999).
7. S. V. Sedov, *Mechanics of Continua* [in Russian], Nauka, Moscow (1984).
8. J. G. Williams and P. D. Ewing, "Fracture under complex stress — the angled crack problem," *Int. J. Fract. Mech.*, **8**, No. 4, 441–446 (1972).
9. J. Tirosh and E. Catz, "Mixed-mode fracture angle and fracture locus of materials subjected to compressive loading," *Eng. Fract. Mech.*, **14**, No. 1, 27–38 (1981).